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ANALYTICAL SOLUTION OF MODIFIED FORCED VAN DER POL VIBRATION EQUATION USING MODIFIED HARMONIC BALANCE METHOD

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Abstract

For obtaining analytical solutions to the modified forced Van der Pol equation, the modified harmonic balance method has been developed. In classical harmonic balance method, the numerical procedure is used for solving a set of nonlinear algebraic equations. But it requires laborious computational attempt and accurate primary guesses values which makes it very hard-working to calculate. According to our method, a set of nonlinear algebraic equations has been converted to a set of linear algebraic equations by using a nonlinear one instead of a set. As a result, it reduces the massive computational work. This method provides not only better results than the existing harmonic balance method but also provides very close solutions to the corresponding numerical results. It is noticeable that there is substantial similarity between the approximate and the numerical results attained by the fourth order Runge-Kutta method. Moreover, the method is facile and straightforward. This technique may play great role to handle strongly nonlinear damped systems with external forces.

Keywords: Harmonic balance method, modified Van der Pol Equation, damped oscillators, forcing term.

Introduction

Nonlinear dynamical systems are important part of human life. Scientists, physicists, applied mathematicians and engineers are forced to find the approximate solutions of the nonlinear physical problems rather than the numerical solutions. A lot of researchers have developed analytical procedures to handle nonlinear oscillatory problems (Alam 2002, 2003, Alam et al. 2006, Lim and Lai 2006, Belendez al. 2012, Guo and Ma 2014, Guo and Leung 2010, Uddin and Sattar 2010, He 1998, Krylov and Bogoliubov 1947, Kovacic and Mickens 2012, Khan 2019, Liu et al. 2007, Mondal et al. 2019, Mishara et al. 2016, Nayfeh 1981, Uddin et al. 2011, 2012, 2015, Ullah et al. 2021a, 2021b, 2021c, 2022, Yeasmin 2020, etc.). Among of them, perturbation techniques (Alam 2002, 2003, Alam et al. 2006, Uddin and Sattar 2010, Krylov and Bogoliubov 1947, Kovacic and Mickens 2012, Nayfeh 1981) are well-established and most popular methods. Perturbation procedures were basically established to solve weakly nonlinear oscillators associated with small parameters.

Numerous approximation procedures have been developed to tackle strongly nonlinear oscillators. These techniques include the modified Lindstedt-Poincare method (Cheung et al. 1991, Liu 2005, Nayfeh 1981), Homotopy perturbation method (He 1998, Uddin et al. 2011, 2021, 2015, Ghosh and Uddin 2021), harmonic balance method (Alam et al. 2016, Lim and Lai 2006, Mickens 1986, Rahman et al. 2010), residual harmonic balance method (Guo and Ma 2014), iterative harmonic balance method (Guo and Leung 2010) etc. in order to describe the action of the heart, Zduniak et al. (2014) created a modified Van der Pol equation with delay. A modified harmonic balance method has recently been developed by Ullah et al. (2021a, 2021b, 2021c and 2022) to manage damped forced Duffing oscillators and forced Van der Pol equation. For handling the modified forced Van der Pol equation, a modified harmonic balance approach has been proposed in this paper. This equation is considered as a description of the normal heart action mode as well as in pathological modes. It is a self-excited oscillator with negative damping which has some external energy sources.

Materials and Methods

Consider the following strongly nonlinear dynamical problem with the term of periodic force is

$$\ddot{x} + \omega_0^2 x + x^3 - \varepsilon g(x, \dot{x}) = p\sin(\omega t) \tag{1}$$

where $x(t)$ is the system's displacement, dots represent the derivative w. r. to t, the natural frequency is ω_0, $g(x, \dot{x})$ is a nonlinear function, $\varepsilon > 0$ is a parameter that need not be small and this parameter stands for the strength of the damping, the amplitude of the external force is p, and the forcing frequency is ω.

The solution of equation (1) is considered to be as per the truncated Fourier series is (Ullah et al. 2021a, 2021b, 2021c, 2022)

$$x = a\cos(\omega t) + b\sin(\omega t) + a_3\cos(3\omega t) + b_3\sin(3\omega t) \tag{2}$$

where a, b, a_3, b_3 are unknown constants. Introducing Eq. (2) in Eq. (1) and comparing the coefficients of equal harmonics, a set of nonlinear algebraic equations are acquired as follows:

$$b(-\omega^2 + \omega_0^2) + a\varepsilon\omega + f_1(a, b, a_3, b_3, \cdots) + \varepsilon S_1(a, b, a_3, b_3, \cdots) = p \tag{3}$$

$$a(-\omega^2 + \omega_0^2) - b\varepsilon\omega + g_1(a, b, a_3, b_3, \cdots) + \varepsilon C_1(a, b, a_3, b_3, \cdots) = 0 \tag{4}$$

$$b_3(-9\omega^2 + \omega_0^2) + 3a_3\varepsilon\omega + f_3(a, b, a_3, b_3, \cdots) + \varepsilon S_3(a, b, a_3, b_3, \cdots) = 0 \tag{5}$$

$$a_3(-9\omega^2 + \omega_0^2) - 3\varepsilon b_3\omega + g_3(a, b, a_3, b_3, \cdots) + \varepsilon C_3(a, b, a_3, b_3, \cdots) = 0 \tag{6}$$

Now using Eq. (3) then excluding ω^2 from the Eqs. (4)- (6) and rejecting the expressions whose effect are small, we attain

$$\omega^2 = \omega_0^2 + f_1(a, b, a_3, b_3, \cdots) + \varepsilon S_1(a, b, a_3, b_3, \cdots) - p/b \tag{7}$$

$$-\varepsilon\beta\omega + \varepsilon C_1(a, b, a_3, b_3, \cdots) - \varepsilon\alpha S_1(a, b, a_3, b_3, \cdots) - af_1(a, b, a_3, b_3, \cdots) \\ + g_1(a, b, a_3, b_3, \cdots) + pa/b = 0 \tag{8}$$

$$-8\omega_0^2\delta - \varepsilon b_3 S_1(a, b, a_3, b_3, \cdots) + \varepsilon S_3(a, b, a_3, b_3, \cdots) - b_3 f_1(a, b, a_3, b_3, \cdots) \\ + f_3(a, b, a_3, b_3, \cdots) + 3\varepsilon a_3\omega + 9pb_3/b = 0 \tag{9}$$

$$-8\omega_0^2 a_3 - \varepsilon a_3(a, b, a_3, b_3, \cdots) + \varepsilon C_3(a, b, a_3, b_3, \cdots) - a_3 f_1(a, b, a_3, b_3, \cdots) \\ + g_3(a, b, a_3, b_3, \cdots) - 3\varepsilon b_3\omega + 9pa_3/b = 0 \tag{10}$$

Now using Eq. (8) and then excluding ω from the Eqs. (9) and (10) and accepting the linear expressions of a_3, b_3 and rejecting the expressions whose effect are small, then the system of linear algebraic equations in a_3, b_3 are acquired. Resolving these equations, a_3, b_3 are formulated associated with a, b. Finally, inserting the values of a_3, b_3 in Eq. (8), and enlarging a as a power series of the small factor $\xi(\varepsilon, \omega, p)$, we obtain

$$a = \mu_0 + \mu_1\xi + \mu_2\xi^2 + \mu_3\xi^3 + \cdots \tag{11}$$

where $\mu_0, \mu_1, \mu_2, \ldots$ are the functions of b. Finally, inserting a, a_3 and b_3 into Eq. (3) and evaluating, b is determined. Successively the assessments of a, a_3 and b_3 are attained.

Example

A modified forced Van der Pol equation is supposed as follows:

$$\ddot{x} + x + x^3 - \varepsilon(1 - x^2)\dot{x} = p\sin(\omega t), \tag{12}$$

where $g(x, \dot{x}) = (1 - x^2)\dot{x}$. The solution of Eq. (12) is considered to be given by the truncated Fourier series is (Ullah et al. 2021a, 2021b, 2021c, 2022)

$$x = a\cos(\omega t) + b\sin(\omega t) + a_3\cos(3\omega t) + b_3\sin(3\omega t) \tag{13}$$

Inserting Eq. (13) into Eq. (12) and comparing the identical harmonic terms, and then deleting the elements that have a minor impact. Then a set of nonlinear equations are formed as follows:

$$4b + 3a^2b + 3b^3 + 4a\varepsilon\omega - a^3\varepsilon\omega - ab^2\varepsilon\omega - 4b\omega^2 + (-6ab - a^2\varepsilon\omega + b^2\varepsilon\omega)a_3 + (3a^2 - 3b^2 - 2ab\varepsilon\omega)b_3 = 4p \tag{14a}$$

$$4a + 3a^3 + 3ab^2 - 4b\varepsilon\omega + a^2b\varepsilon\omega + b^3\varepsilon\omega - 4a\omega^2 + (3a^2 - 3b^2 - 2ab\varepsilon\omega)a_3 + (6ab + a^2\varepsilon\omega - b^2\varepsilon\omega)b_3 = 0 \tag{14b}$$

$$3a^2b - b^3 - a^3\varepsilon\omega + 3ab^2\varepsilon\omega - 3\varepsilon\omega a_3^3 + (4 + 6a^2 + 6b^2 - 36\omega^2)b_3 + 3a_3^2b_3 - 3\varepsilon\omega a_3\big(2(-2 + a^2 + b^2)\big) = 0 \tag{14c}$$

$$a^3 - 3ab^2 + 3a^2b\varepsilon\omega - b^3\varepsilon\omega + 6(-2 + a^2 + b^2)\varepsilon\omega b_3 + a_3(4 + 6a^2 + 6b^2 - 36\omega^2) = 0 \tag{14d}$$

Terminating ω^2 from Eqs. (14b)- (14d) with help of Eq. (14a) and deleting the elements that have a minor impact, we get

$$ap - 4a^2\varepsilon\omega + a^4\varepsilon\omega - 4b^2\varepsilon\omega + 2a^2b^2\varepsilon\omega + b^4\varepsilon\omega + 9a^2ba_3 - 3b^3a_3 + a^3\varepsilon\omega a_3 - 3ab^2\varepsilon\omega a_3 - 3a^3b_3 + 9ab^2b_3 + 3a^2b\varepsilon\omega b_3 - b^3\varepsilon\omega b_3 = 0 \tag{15a}$$

$$3a^2b^2 - b^4 - a^3b\varepsilon\omega + 3ab^3\varepsilon\omega + 12b\varepsilon\omega a_3 - 6a^2b\varepsilon\omega a_3 - 6b^3\varepsilon\omega a_3 - 32bb_3 - 21a^2bb_3 - 21b^3b_3 + 36pb_3 - 36a\varepsilon\omega b_3 + 9a^3\varepsilon\omega b_3 + 9ab^2\varepsilon\omega b_3 = 0 \tag{15b}$$

$$a^3b - 3ab^3 + 3a^2b^2\varepsilon\omega - b^4\varepsilon\omega - 32ba_3 - 21a^2ba_3 - 21b^3a_3 + 36pa_3 - 36a\varepsilon\omega a_3 + 9a^3\varepsilon\omega a_3 + 9ab^2\varepsilon\omega a_3 - 12b\varepsilon\omega b_3 + 6a^2b\varepsilon\omega b_3 + 6b^3\varepsilon\omega b_3 = 0 \tag{15c}$$

Now excluding ω from the Eqs. (15b) and (15c) using Eq. (15a), and assuming the linear terms of a_3, b_3 and rejecting the terms that have a minor impact, we attain

$$-12a^4b^2 + 3a^6b^2 - 8a^2b^4 + 5a^4b^4 + 4b^6 + a^2b^6 - b^8 + 4a^4bp - 12a^2b^3p + 128a^2bb_3 + 52a^4bb_3 - 24a^6bb_3 + 128b^3b_3 + 104a^2b^3b_3 - 36a^4b^3b_3 + 52b^5b_3 - 96a^2b^5b_3 - 20b^7b_3 - 144b^2pb_3 + 36a^2b^2pb_3 + 36b^4pb_3 = 0 \tag{16a}$$

$$-4a^5b + a^7b + 8a^3b^3 - a^5b^3 + 12ab^5 - 5a^3b^5 - 3ab^7 - 12a^3b^2p + 4ab^4p + 128a^2ba_3 + 52a^4ba_3 - 20a^6ba_3 + 128b^3a_3 + 104a^2b^3a_3 - 96a^4b^3a_3 + 52b^5a_3 - 36a^2b^5a_3 - 24b^7a_3 - 144b^2pa_3 + 36a^2b^2pa_3 + 36b^4pa_3 = 0 \tag{16b}$$

Solving Eqs. (16a) and (16b), a_3 and b_3 are calculated as:

$$a_3 = \frac{l_1}{m_1}, \qquad b_3 = \frac{l_2}{m_1} \tag{17a}$$

where

$$l_1 = -4a^5 + a^7 + 8a^3 b^2 - a^5 b^2 + 12ab^4 - 5a^3 b^4 - 3ab^6 - 12a^3 bp + 4ab^3 p \tag{17b}$$
$$l_2 = -12a^4 b + 3a^6 b - 8a^2 b^3 + 5a^4 b^3 + 4b^5 + a^2 b^5 - b^7 + 4a^4 p - \qquad 12a^2 b^2 p \tag{17c}$$
$$m_1 = 4(-32a^2 - 13a^4 + 6a^6 - 32b^2 - 26a^2 b^2 + 9a^4 b^2 - 13b^4 + \qquad 24a^2 b^4 + 5b^6 + \tag{17d}$$
$$36bp - 9a^2 bp - 9b^3 p)$$

Putting a_3 and b_3 into Eq. (15a) and enlarging a as a power series of the small parameter ξ, we have

$$a = \mu_0 + \mu_1 \xi + \mu_2 \xi^2 + \mu_3 \xi^3 \tag{18a}$$

where

$$\xi = \frac{\varepsilon \omega}{p}, \mu_0 = \frac{b^2 \varepsilon \omega}{p}, \mu_1 = \frac{b^4 \varepsilon^2 \omega^2}{p^2}, \mu_2 = \frac{2b^6 \varepsilon^3 \omega^3}{p^3}, \mu_3 = \frac{5b^8 \varepsilon^4 \omega^4}{p^4} \tag{18b}$$

Finally, the values of b are obtained by replacing the values of a, a_3 and b_3 into Eq. (14a). Successively, the values of a, a_3 and b_3 are attained. Thus, the complete solution of Eq. (12) is obtained.

Results

To demonstrate the strength and reliability of the present procedure, the numerical results attained by the well-known Runge-Kutta method are compared to those results determined by the present technique in Table 1 and in Figures 1(a)–1(d) graphically. The phase planes are also shown in Figures 2(a)-2(d). These Figures clarify how the dynamical systems behave. All of these Figures show that the attained results comply with those outcomes getting by using the numerical technique. Table 1 shows that the results attained by the developed technique and the numerical technique exhibit good congruence.

Table 1. A comparison within the results acquired by the present method and the numerical method

| Time, t	$\omega = 12, \quad \varepsilon = 0.5, \quad p = 15$		$\omega = 10, \quad \varepsilon = 1, \quad p = 20$	
	Approximate Solution, x_{app}	Numerical Solution, x_{nu}	Approximate Solution, x_{app}	Numerical Solution, x_{nu}
0.0	0.004	0.004	0.020	0.020
0.5	0.033	0.008	0.198	0.201
1.0	0.060	0.060	0.092	0.092
1.5	0.082	0.094	-0.145	-0.169
2.0	0.097	0.097	-0.174	-0.174
2.5	0.104	0.103	0.047	0.084
3.0	0.103	0.103	0.201	0.200
3.5	0.094	0.081	0.067	0.027
4.0	0.078	0.078	-0.163	-0.164
4.5	0.055	0.032	-0.160	-0.182
5.0	0.028	0.028	0.072	0.070
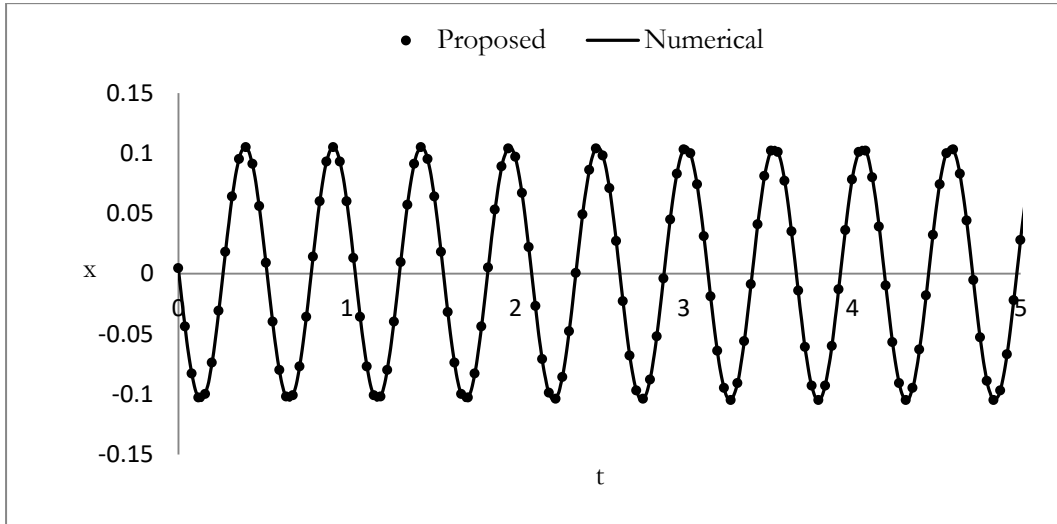
Figure 1(a). The proposed method (shown as dots) and the numerical technique are used to compare the time versus displacement of Eq. (12) when $\omega = 12,\ \varepsilon = 0.5,\ p = 15$.
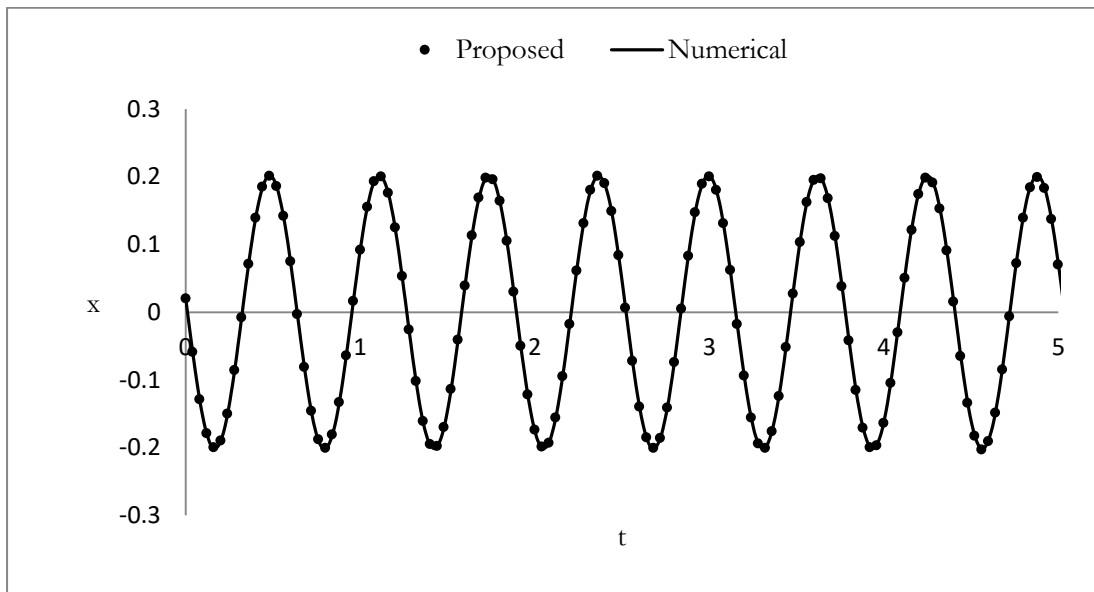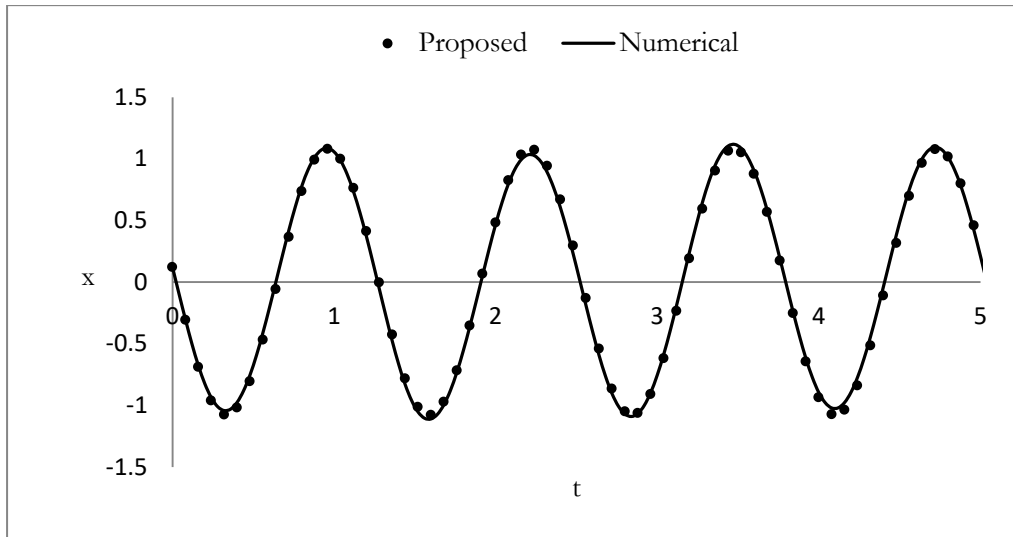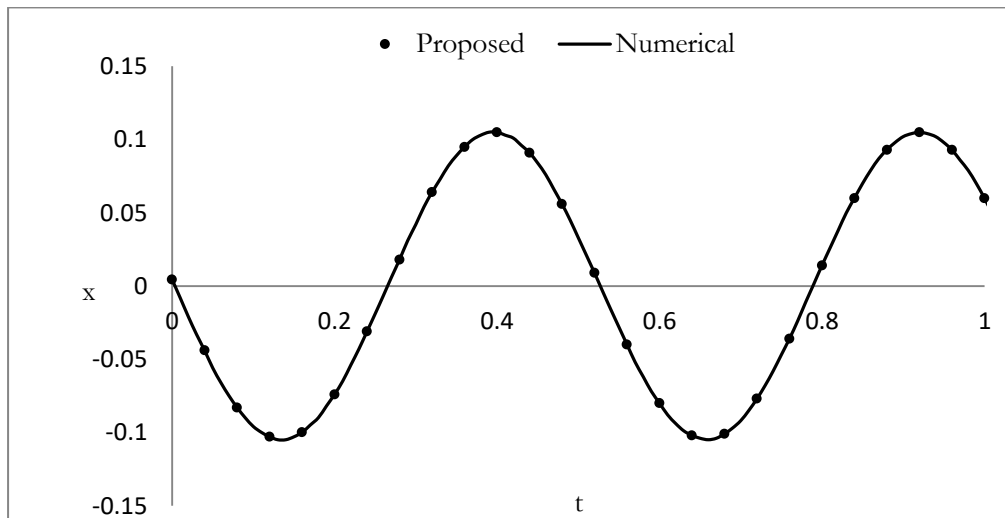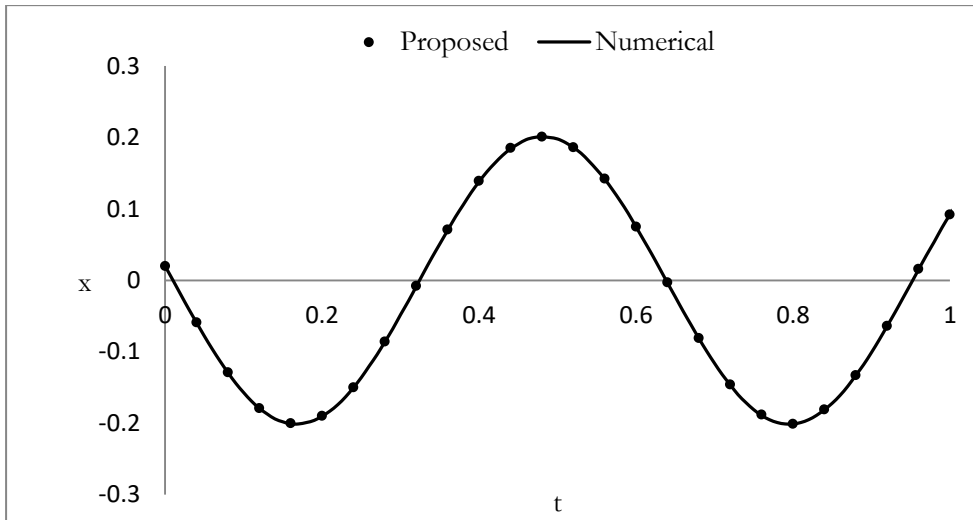

Figure 1(b). The proposed method (shown as dots) and the numerical technique are used to compare the time versus displacement of Eq. (12) when $\omega = 10,\ \varepsilon = 1.0,\ p = 20$.

Figure 1(c). The proposed method (shown as dots) and the numerical technique are used to compare the time versus displacement of Eq. (12) when $\omega = 5, \; \varepsilon = 0.5, \; P = 25$.
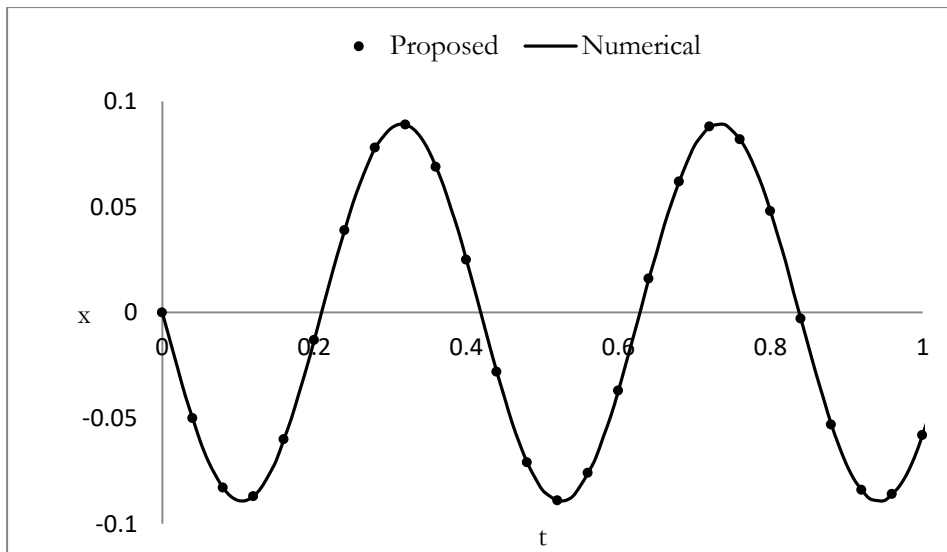

Figure 1(d). The proposed method (shown as dots) and the numerical technique are used to compare the time versus displacement of Eq. (12) when $\omega = 12, \; \varepsilon = 0.5, \; p = 15$.

Figure 1(e). The proposed method (shown as dots) and the numerical technique are used to compare the time versus displacement of Eq. (12) when $\omega = 10, \ \varepsilon = 1.0, \ p = 20$.

Figure 1(f). The proposed method (shown as dots) and the numerical technique are used to compare the time versus displacement of Eq. (12) when $\omega = 15, \ \varepsilon = 0.01, \ p = 20$.
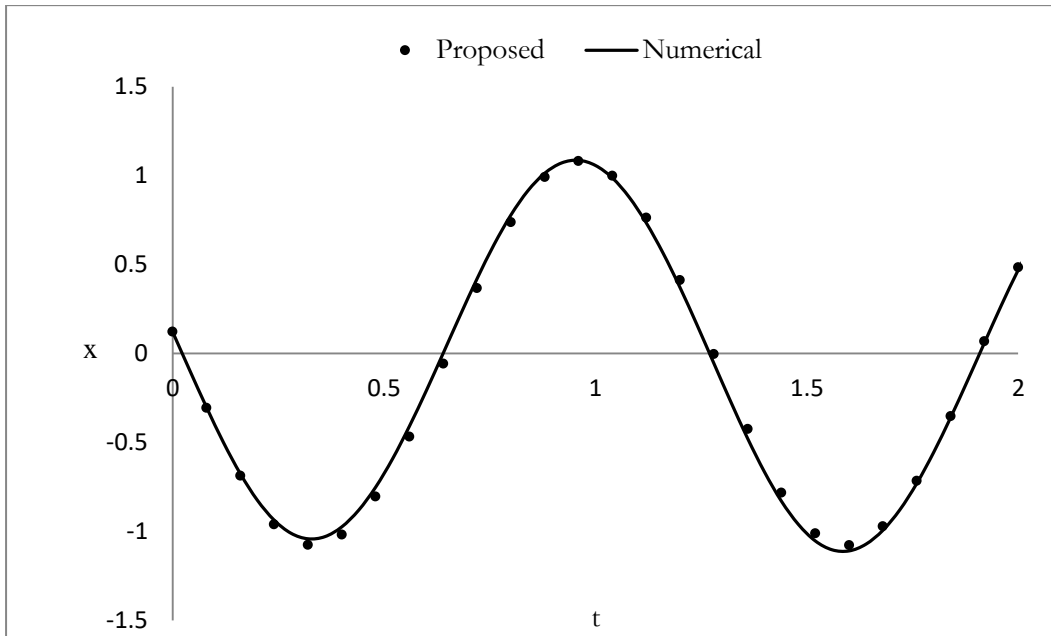
Figure 1(g). The proposed method (shown as dots) and the numerical technique are used to compare the time versus displacement of Eq. (12) when $\omega = 5, \; \varepsilon = 0.5, \; P = 25$.
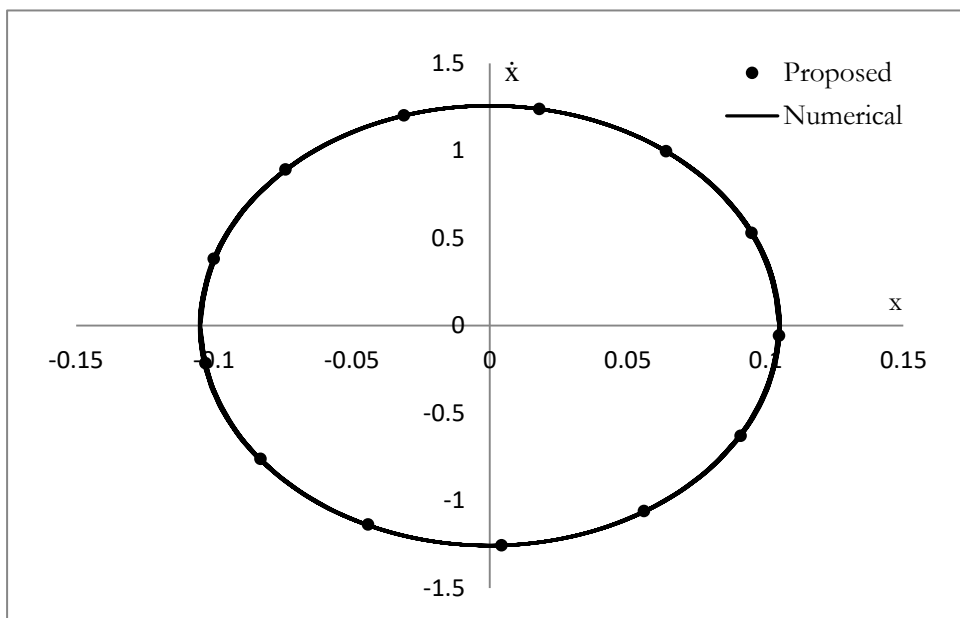

Figure 2(a). Analytical and the numerical results are compared in the phase plane for $\omega = 12, \; \varepsilon = 0.5, \; P = 15$.
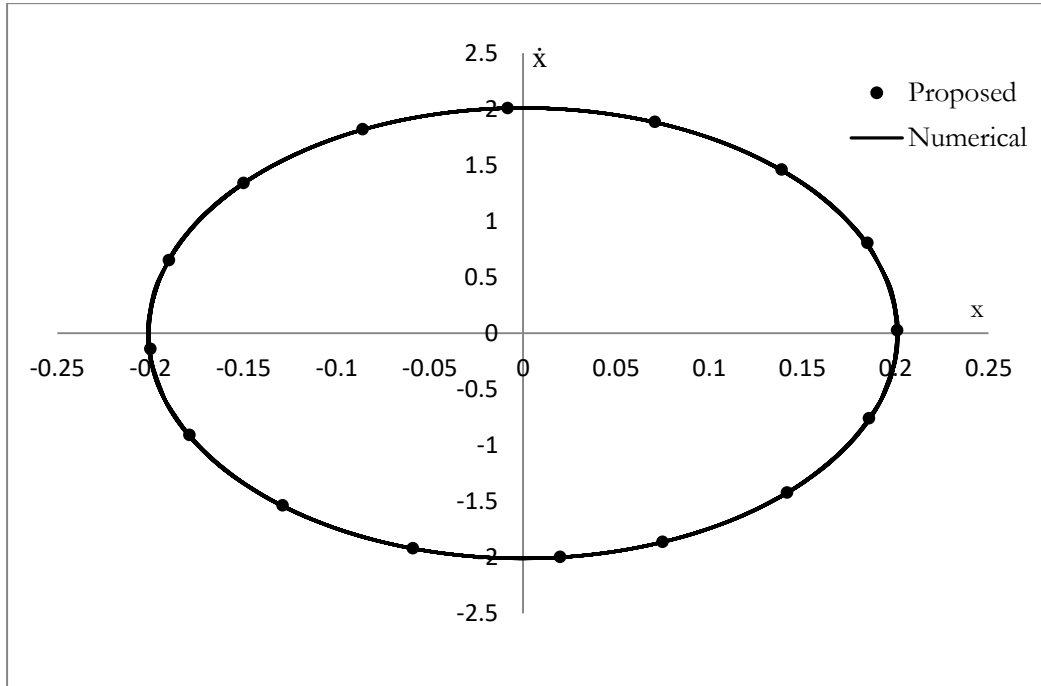
Figure 2(b). Analytical and the numerical results are compared in the phase plane for $\omega = 10, \ \varepsilon = 1.0, \ P = 20$.
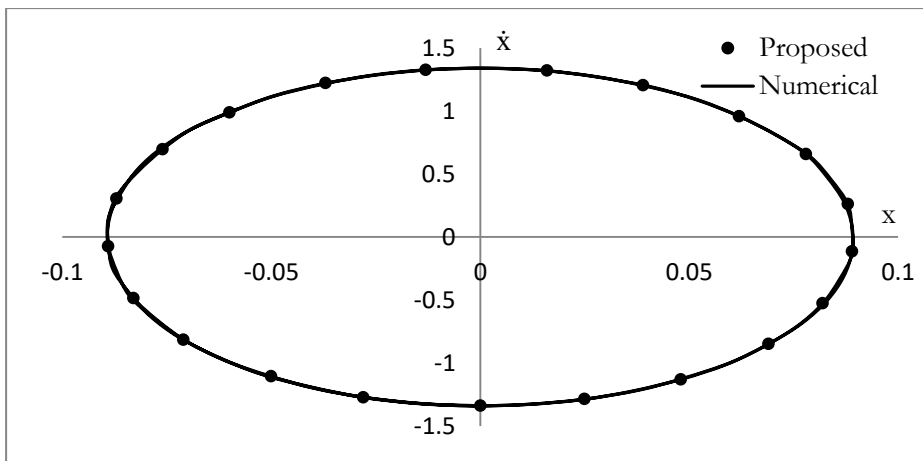

Figure 2(c). Analytical and the numerical results are compared in the phase plane for $\omega = 15, \ \varepsilon = 0.01, \ P = 20$.
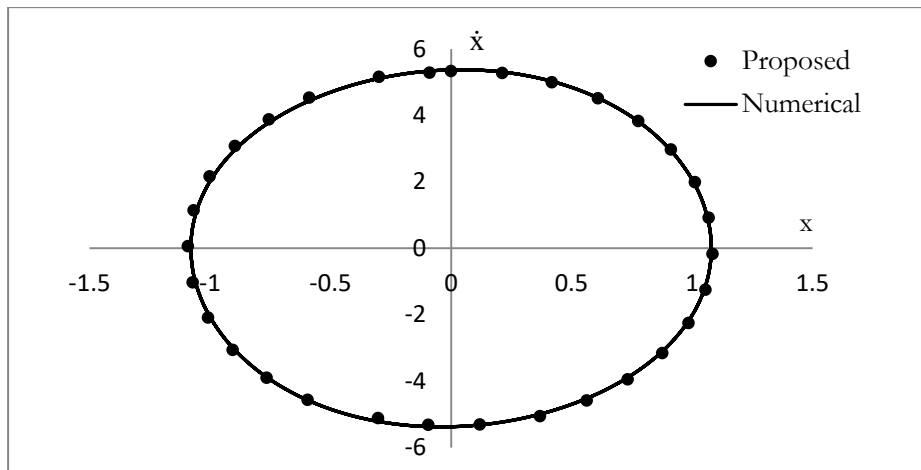
Figure 2(d). Analytical and the numerical results are compared in the phase plane for $\omega = 5$, $\varepsilon = 0.5$, $P = 25$.

Discussion

Determination of the analytical solutions of the nonlinear dynamical problems is very difficult for the presence of strong nonlinearity. There are no common well-established procedures to handle nonlinear dynamical problems. Several methods have been developed to solve particular nonlinear dynamical problems. The benefit of the present method is that it can transform a set of nonlinear algebraic equations into a set of linear algebraic equations by consuming a nonlinear one against of a set. As a result, it has been required less computational work to estimate the values of the unknown constants more than the other existing classical harmonic balance method. It should be noted that the outcomes of the proposed method exhibit an acceptable degree of closeness to the numerical solutions obtained by using Runge-Kutta method for large damping and various values of the parameters. From Table 1 and Figures, it is noticeable that there is substantial agreement between the approximate and the numerical answers. It is hard to solve nonlinear dynamical systems by considering the large number of harmonics in a trial solution.

Although numerical algorithms are very simple to develop, they require significant computing work and accurate beginning predictions. Additionally, these procedures do not have any analytical expressions besides results. On the other hand, scientists, applied mathematicians, physicists, engineers, and researchers favor analytical approximation approaches. Since these techniques have logical expressions which are essential in physical meanings and additional worthy of parametric study. The behaviors of the oscillating processes are characterized by the amplitude and phase variables of the dynamical systems. Hence graphical representation for any physical problem is very important to the scientists, applied mathematicians, physicists, engineers and researchers. In this paper, we try to contribute a little in this field.

Conclusion

It is mentioned that the proposed method is very efficient and more suitable for solving strongly nonlinear modified forced Van der Pol oscillators in the whole solution domain. The outcomes show that the proposed method is competent and time-saving method. Therefore, this technique may be considered as a suitable technique for handling damped nonlinear physical problems in the presence of periodic external force.

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